Received: 14 December 2018 | Revised: 6 March 2019 | Accepted: 11 March 2019

DOI: 10.1002/mana.201800546

MATHEMATISCHE NACHRICHTEN

ORIGINAL PAPER

Stability of Timoshenko systems with thermal coupling on the bending moment

C. L. Cardozo[1] | **M. A. Jorge Silva**[1] | **T. F. Ma**[2] | **J. E. Muñoz Rivera**[3,4]

[1]Department of Mathematics, State University of Londrina, 86057-970 Londrina, PR, Brazil

[2]Institute of Mathematical and Computer Sciences, University of São Paulo, 13566-590 São Carlos, SP, Brazil

[3]National Laboratory of Scientific Computation, 25651-070 Petrópolis, RJ, Brazil

[4]Institute of Mathematics, Federal University of Rio de Janeiro, 21941-909 Rio de Janeiro, RJ, Brazil

Correspondnce

M. A. Jorge Silva, Department of Mathematics, State University of Londrina, 86057-970 Londrina, PR, Brazil.

Email: marcioajs@uel.br

Funding information

Conselho Nacional de Desenvolvimento Científico e Tecnológico, Grant/Award Numbers: 441414/2014-1, 308837/2014-2, 310041/2015-5, 402689/2012-7

Abstract

The Timoshenko system is a distinguished coupled pair of differential equations arising in mathematical elasticity. In the case of constant coefficients, if a damping is added in only one of its equations, it is well-known that exponential stability holds if and only if the wave speeds of both equations are equal. In the present paper we study both non-homogeneous and homogeneous thermoelastic problems where the model's coefficients are non-constant and constants, respectively. Our main stability results are proved by means of a unified approach that combines local estimates of the resolvent equation in the semigroup framework with a recent control-observability analysis for static systems. Therefore, our results complement all those on the linear case provided in [22], by extending the methodology employed in [4] to the case of Timoshenko systems with thermal coupling on the bending moment.

KEYWORDS

exponential stability, polynomial decay, thermoelasticity, Timoshenko system

MSC (2010)

35B35, 35B40, 35M33, 74D05, 74K10

1 | INTRODUCTION

The Timoshenko system is a widely accepted model for vibrations of elastic beams. Its mathematical formulation is given by a system of two partial differential equations

$$\rho A \varphi_{tt} = S_x \quad \text{and} \quad \rho I \psi_{tt} = M_x - S, \tag{1.1}$$

where, $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$ stand for the transversal displacement and rotation angle, respectively. In its rest configuration the beam coincides with an interval of the x-axis, here denoted by $[0,l]$. The quantities M and S represent, respectively, the bending moment and the shear stress. For the remaining constants, ρ is the mass density and A and I denote the area and the inertial moment of the transversal section, respectively. The corresponding constitutive elastic laws are given by

$$S = \kappa A(\varphi_x + \psi) \quad \text{and} \quad M = EI\psi_x, \tag{1.2}$$

where EI represents the flexural rigidity of the material and κ is a shear coefficient. Then one obtains from (1.1) and (1.2) the classical Timoshenko system

$$\rho_1\varphi_{tt} - (k(\varphi_x + \psi))_x = 0, \tag{1.3}$$

$$\rho_2\psi_{tt} - (b\,\psi_x)_x + k(\varphi_x + \psi) = 0, \tag{1.4}$$

where $\rho_1 = \rho A, \rho_2 = \rho I, k = \kappa A, b = EI$. See for instance [31, Section 55].

In this paper we are interested in the asymptotic stability of the system when a damping is added only in the equation for rotation angle. In this direction, a remarkable result was given by Soufyane [30], which asserts that the Timoshenko system with a damping $-\beta(x)\psi_t$ added in Equation (1.4) is exponentially stable, if and only if,

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}. \tag{1.5}$$

This means that the energy of the system decays exponentially if and only if the wave speeds of both Equations (1.3)–(1.4) are equal. In this case there is an effective transfer of the dissipation from the damped equation to the undamped one. The condition (1.5) was later used and extended to Timoshenko systems with other type of dissipations or forcing terms. See e.g. [1,5,9–11,13,14,17,20,28,29]. However, condition (1.5) is not needed to stabilize exponentially the system when both equations are damped. Indeed, this situation was studied, for instance, in [8,18,27].

On the other hand, we observe that the above mentioned works are all concerned with the Timoshenko system with constant coefficients. With respect to non-homogeneous coefficients, the partially damped Timoshenko system was studied by Ammar-Khodja, Kerbal and Soufyane [6]. They considered internal and boundary dissipations acting only in the equation for rotation angle (1.4) and proved that exponential stability holds if, as a function of x, condition (1.5) is satisfied for all $x \in [0, l]$. However, in practice, it is very unlikely that both equations in the non-homogeneous Timoshenko system have equal wave speeds for all $x \in [0, l]$. Indeed, for constant coefficients, condition (1.5) is never achieved, as observed in [24].

Here, in order to achieve (uniform) exponential stability, our first main result (Theorem 2.2) proposes a more flexible and general condition than asking (1.5) for all $x \in [0, l]$. Roughly speaking, for the partially damped thermoelastic system, we prove that exponential stability holds provided that condition (1.5) is satisfied locally, that is, for any small enough open sub-interval $I \subset [0, l]$, we just assume that

$$\frac{\rho_1(x)}{k(x)} = \frac{\rho_2(x)}{b(x)} \quad \text{for} \quad x \in I. \tag{1.6}$$

Otherwise, only polynomial decay is proved with rate depending on the regularity of initial data (Theorem 2.3). Such stability results are precisely clarified in Section 2 to a new non-homogeneous thermoelastic system (non-constant coefficients), where by following [22] we assume that a thermal dissipation is applied on the bending moment

$$M_\theta = b\psi_x - m\theta, \tag{1.7}$$

where θ denotes the relative temperature and $m(x) > 0$ is a non-constant coupling coefficient. In addition, the homogeneous thermoelastic system (constant coefficients) previously considered in [22] is also approached in Section 5 with some different boundary conditions. In such a case, and working in the general case where (1.5) is not taken into account, our main result is Theorem 3.2 that proves the same polynomial decay independent of boundary conditions. Comparisons with existing results on the subject are given as follows as well as how the remaining work is organized.

(i) The non-homogenous thermoelastic Timoshenko problem (see (2.1)–(2.5) in Section 2) is first addressed in the present article. Theorems 2.2 and 2.3 are new and have been proved by relying on a careful control-observability analysis, which is recalled in the Appendix A for the sake of the reader. It is worth mentioning that, as far as we know, the locally equal wave speeds assumption (1.6) has not been previously considered in the literature for non-homogeneous Timoshenko systems with thermal coupling on the bending moment under the Fourier law. Therefore, Theorem 2.2 is the first main result of the section. Moreover, even the case where (1.6) is not assumed, then Theorem 2.3 provides a unified approach on polynomial stability that was not considered previously in [22] even for constant coefficients.

(ii) Although the homogeneous thermoelastic Timoshenko system (see (3.1)–(3.5) in Section 5) has already been addressed in [22], our stability results in this section provide a significantly complement of those contained therein to the linear case. Indeed, in this part the main result is Theorem 3.2 where one proves that the problem (3.1)–(3.5) is, in general, polynomially stable with decay rate depending on the regularity of initial data, but independently of boundary conditions considered in (3.5). In addition, Theorem 3.3 also asserts on optimality in some specific cases. Such statements were not addressed in [22] when (1.5) does not hold. Even in the case where assumption (1.5) is regarded, Theorem 3.1 complements the corresponding result in [22] because it gives a unified approach on exponential stability independently of boundary conditions through the semigroup theory, which differentiates of the approach in [22, Sects. 2 and 3], where perturbation energy methods are employed for each boundary condition separately.

(iii) In spite of using the same tools as in [3,4] to reach our stability results in Sections 2 and 5, we observe that the proofs of them rely on totally different process, by requiring new (and distinct) estimates. In fact, by following the thermal law applied on the shear force as in [2], the authors in [3,4] consider

$$S_\theta = k(\varphi_x + \psi) - m\theta, \tag{1.8}$$

instead of (1.7). Then, the propagation of dissipativity to the whole terms of the solution is given as in the diagram (see e.g. [4, Subsect. 3.1]):

$$\boxed{\theta \quad \xmapsto{\text{1st}} \quad S = k(\varphi_x + \psi) \quad \xmapsto{\text{2nd}} \quad M = b\psi_x}$$

whereas in the present work such way for the estimates are not useful. Here the picture is different because the propagation of the dissipative-estimates walks on the next opposite diagram

$$\boxed{\theta \quad \xmapsto{\text{1st}} \quad M = b\psi_x \quad \xmapsto{\text{2nd}} \quad S = k(\varphi_x + \psi)}$$

as clarified in the computations of Subsection 2.3.

(iv) Last, we would like to remark that a comparison with thermoelastic Timoshenko systems that are governed by different thermal laws like Cattaneo, Gurtin-Pipkin, Type III, among others, as given e.g. in [11,13,14,29], is more delicate due to the different character of such systems. However, we believe that the approach on local estimates provided in Section 2 along with the observability result in the Appendix A can be probably applied to these other thermoelastic systems taking into account different constitutive laws.

2 | NON-HOMOGENEOUS THERMOELASTIC SYSTEM

In this section, for $l > 0$ and $\mathbb{R}^+ = (0, \infty)$, we consider the following initial-boundary value problem with non-constant coefficients

$$\rho_1\varphi_{tt} - (k(\varphi_x + \psi))_x = 0 \quad \text{in} \quad (0, l) \times \mathbb{R}^+, \tag{2.1}$$

$$\rho_2\psi_{tt} - (b\psi_x)_x + k(\varphi_x + \psi) + (m\theta)_x = 0 \quad \text{in} \quad (0, l) \times \mathbb{R}^+, \tag{2.2}$$

$$\rho_3\theta_t - (c\,\theta_x)_x + m\psi_{xt} = 0 \quad \text{in} \quad (0, l) \times \mathbb{R}^+, \tag{2.3}$$

$$(\varphi(\cdot, 0), \varphi_t(\cdot, 0), \psi(\cdot, 0), \psi_t(\cdot, 0), \theta(\cdot, 0)) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0) \ \text{in} \ (0, l), \tag{2.4}$$

$$\varphi(0, t) = \varphi(l, t) = \psi(0, t) = \psi(l, t) = \theta(0, t) = \theta(l, t) = 0, \ \ t \geq 0, \tag{2.5}$$

where $\rho_1, \rho_2, \rho_3, k, b, c, m$ are functions satisfying

$$\rho_1, \rho_2, \rho_3, k, b, c, m \in W^{1,\infty}(0, l), \ \ \rho_1, \rho_2, \rho_3, k, b, c, m > 0 \ \text{in} \ (0, l). \tag{2.6}$$

The weak phase space $\mathcal{H}$ for solutions of (2.1)–(2.5) is defined by

$$\mathcal{H} = H_0^1(0, l) \times L^2(0, l) \times H_0^1(0, l) \times L^2(0, l) \times L^2(0, l),$$

endowed with inner-product and norm

$$\left(U, \tilde{U}\right)_{\mathcal{H}} = \int_0^l \left[\rho_1\Phi\overline{\tilde{\Phi}} + \rho_2\Psi\overline{\tilde{\Psi}} + b\psi_x\overline{\tilde{\psi}_x} + k(\varphi_x + \psi)\overline{(\tilde{\varphi}_x + \tilde{\psi})} + \rho_3\theta\overline{\tilde{\theta}}\right] dx, \tag{2.7}$$

$$\|U\|_{\mathcal{H}}^2 = \int_0^l \left[\rho_1|\Phi|^2 + \rho_2|\Psi|^2 + b|\psi_x|^2 + k|\varphi_x + \psi|^2 + \rho_3|\theta|^2\right] dx, \tag{2.8}$$

for all $U = (\varphi, \Phi, \psi, \Psi, \theta), \tilde{U} = \left(\tilde{\varphi}, \tilde{\Phi}, \tilde{\psi}, \tilde{\Psi}, \tilde{\theta}\right) \in \mathcal{H}$, which turns $\mathcal{H}$ into a Hilbert space under the assumption (2.6) on the coefficients.

2.1 | Semigroup solution

Denoting $\varphi_t = \Phi, \psi_t = \Psi$ and $U = (\varphi, \Phi, \psi, \Psi, \theta)$, then the system (2.1)–(2.5) can be written as the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}U = \mathcal{A}U, \quad t > 0, \\ U(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0) := U_0, \end{cases} \tag{2.9}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}U = \begin{pmatrix} \Phi \\ \dfrac{1}{\rho_1}(k(\varphi_x + \psi))_x \\ \Psi \\ \dfrac{1}{\rho_2}\left[(b\,\psi_x)_x - k(\varphi_x + \psi) - (m\,\theta)_x\right] \\ \dfrac{1}{\rho_3}\left[(c\,\theta_x)_x - m\Psi_x\right] \end{pmatrix}, \quad U \in D(\mathcal{A}), \tag{2.10}$$

with domain

$$D(\mathcal{A}) = \left(H^2 \cap H_0^1\right)(0,l) \times H_0^1(0,l) \times \left(H^2 \cap H_0^1\right)(0,l) \times H_0^1(0,l) \times \left(H^2 \cap H_0^1\right)(0,l).$$

The existence and uniqueness result to the Cauchy problem (2.9), and consequently to the equivalent system (2.1)–(2.5), reads as follows:

Theorem 2.1. *Let us assume that condition* (2.6) *holds.*

(i) If $U_0 \in \mathcal{H}$, then the problem (2.9) *has a unique mild solution in the class*

$$U \in C^0([0,\infty), \mathcal{H}).$$

(ii) If $U_0 \in D(\mathcal{A})$, then the above mild solution is regular one and satisfies

$$U \in C^0([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H}).$$

(iii) If $U_0 \in D(\mathcal{A}^n)$, $n \geq 2$ integer, then the above regular solution satisfies

$$U \in \bigcap_{j=0}^{n} C^{n-j}\left([0,+\infty), D\left(\mathcal{A}^j\right)\right).$$

Proof. It is not difficult to check that $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ stands for the resolvent set of $\mathcal{A}$, and a straightforward computation shows that $\mathcal{A}$ is dissipative with

$$\operatorname{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -\int_0^l c(x)|\theta_x(x)|^2\,dx \leq 0, \quad U \in D(\mathcal{A}). \tag{2.11}$$

As a consequence of the Lummer–Philips theorem (cf. [25]), $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t) = e^{\mathcal{A}t}$ defined in $\mathcal{H}$. Therefore, the solution of (2.9) satisfying items (i)–(iii) is given by

$$U(t) = e^{\mathcal{A}t}U_0, \quad t \geq 0.$$

□

2.2 | Stability results

Next we are going to present our results on stability to the solution $U(t) = e^{\mathcal{A}t}U_0$ of the problem (2.9), and consequently to the solution of the system (2.1)–(2.5). Before doing so, we note that to the exponential stability result we assume a local condition on equal wave speeds like

$$\frac{k(x)}{\rho_1(x)} = \frac{b(x)}{\rho_2(x)}, \quad \text{for } x \text{ in any open subinterval } \ I \subset (0,l). \tag{2.12}$$

In this section our main stability results are the following:

Theorem 2.2 (Exponential Stability). *Let us assume that conditions* (2.6) *and* (2.12) *hold. Then, there exist positive constants* $C, \gamma > 0$*, independent of the initial data* $U_0 \in \mathcal{H}$*, such that*

$$\|U(t)\|_{\mathcal{H}} \leq Ce^{-\gamma t}\|U_0\|_{\mathcal{H}}, \quad t > 0. \tag{2.13}$$

In other words, the non-homogeneous thermoelastic Timoshenko system (2.1)–(2.5) *is (uniformly) exponentially stable.*

Theorem 2.3 (Polynomial Stability). *Let us only suppose that condition* (2.6) *holds. Then, for every integer* $n \geq 1$*, there exists a constant* $C_n > 0$*, independent of the initial data* $U_0 \in D(\mathcal{A}^n)$*, such that*

$$\|U(t)\|_{\mathcal{H}} \leq \frac{C_n}{t^{n/2}}\|U_0\|_{D(\mathcal{A}^n)}, \quad t \to +\infty. \tag{2.14}$$

In other words, the system (2.1)–(2.5) *is (non-uniformly) polynomially stable with rate depending on the regularity of initial data.*

The proof of Theorems 2.2 and 2.3 will be conclude later as a consequence of some technical lemmas proved below together with an observability result for static Timoshenko-type systems, which is presented in the Appendix A, and both combined with some abstract results in the theory of linear semigroups that can be found in [7,12,15,19,26]. For didactic reasons we are going to remember such abstract results in the next subsection, but they can be skipped by the readers that are familiar with the subject.

2.2.1 | Theoretical results in spectral theory for linear semigroups

The first result is a classical one in the theory of linear semigroups. It characterizes the exponential stability of C_0-semigroups, see for instance [15,26]. For the sake of simplicity, we recall it by following the presentation to Hilbert spaces, see e.g. Liu and Zheng [19].

Theorem 2.4 ([19, Theorem 1.3.2]). *A* C_0*-semigroup of contractions* $T(t) = e^{At}$ *over a Hilbert space* H *is exponentially stable if and only if*

$$i\mathbb{R} \subseteq \rho(A) \quad \text{and} \quad \limsup_{|\beta|\to\infty} \left\| \left(i\beta I_d - A\right)^{-1} \right\|_{\mathcal{L}(H)} < \infty, \tag{2.15}$$

where $i\mathbb{R} = \{i\beta \mid \beta \in \mathbb{R}\}$.

The second result is a summarized version of the general one established by Borichev and Tomilov [7]. It provides somehow a characterization of polynomial stability for bounded C_0-semigroups defined on Hilbert spaces.

Theorem 2.5 ([7, Theorem 2.4]). *Let* $T(t) = e^{At}$ *be a bounded* C_0*-semigroup on a Hilbert space* H *such that* $i\mathbb{R} \subset \rho(A)$*. Then,*

$$\left\|T(t)A^{-1}\right\|_{\mathcal{L}(H)} = O\left(t^{-1/\lambda}\right) \quad \text{if and only if} \quad \left\|\left(i\beta I_d - A\right)^{-1}\right\|_{\mathcal{L}(H)} = O\left(|\beta|^{\lambda}\right), \tag{2.16}$$

for $t \to \infty$ *and* $|\beta| \to \infty$*, and some fixed constant* $\lambda > 0$.

Lastly, we observe that the typical condition $i\mathbb{R} \subset \rho(A)$ usually plays an important role in the stabilization theory for linear C_0-semigroups. Indeed, it is worth mentioning that such condition is required in both Theorems 2.4 and 2.5. Therefore, by following Engel and Nagel's book [12], we briefly remind in the next result that it is possible to characterize the spectrum $\sigma(A) = \mathbb{C}\backslash\rho(A)$ of linear operators $A : D(A) \subset X \to X$ whose domain $D(A)$ is compactly embedded in a Banach space X.

Theorem 2.6 ([12, Proposition 5.8 and Corollary 1.15]). *Let* $\left(X, \|\cdot\|_X\right)$ *be a Banach space and consider* $A : D(A) \subset X \to X$ *a linear operator with nonempty resolvent set* $\rho(A)$.

I. *A has compact resolvent (that is, there exists $\lambda \in \rho(A)$ such that $(\lambda I - A)^{-1}$ is compact) if and only if the canonical injection $i : \big(D(A), \|\cdot\|_{D(A)}\big) \to \big(X, \|\cdot\|_X\big)$ is compact.*

II. *If the operator A has compact resolvent, then the spectrum $\sigma(A)$ consists only of eigenvalues of A.*

2.3 | Proofs of the main results: Theorems 2.2 and 2.3

We start by considering the resolvent equation

$$i\beta U - \mathcal{A}U = F, \tag{2.17}$$

where $U = (\varphi, \Phi, \psi, \Psi, \theta)$, $F = (f_1, f_2, f_3, f_4, f_5)$ and $\mathcal{A}$ is defined in (2.10), and its component equations

$$i\beta\varphi - \Phi = f_1, \tag{2.18}$$

$$i\beta\rho_1\Phi - (k(\varphi_x + \psi))_x = \rho_1 f_2, \tag{2.19}$$

$$i\beta\psi - \Psi = f_3, \tag{2.20}$$

$$i\beta\rho_2\Psi - (b\,\psi_x)_x + k(\varphi_x + \psi) + (m\,\theta)_x = \rho_2 f_4, \tag{2.21}$$

$$i\beta\rho_3\theta - (c\,\theta_x)_x + m\Psi_x = \rho_3 f_5. \tag{2.22}$$

We also observe that the assumption (2.6) implies the existence of constants $c_0, c_1 > 0$ such that

$$c_0 \le \rho_1,\ \rho_2,\ \rho_3,\ k,\ b,\ c,\ m \le c_1 \ \text{ and } \ \rho_1',\ \rho_2',\ \rho_3',\ k',\ b',\ c',\ m' \le c_1 \ \text{ on } \ (0, l). \tag{2.23}$$

Hereafter, to simplify the notations, we shall use the same parameter $C > 0$ to denote several different positive constants in forthcoming computations and, as usual, $\|\cdot\|_{L^p}$ stands for the norm in $L^p(0, l)$. Besides, the well-known Hölder and Poincaré's inequalities will be constantly used in the estimates so that we allowed ourselves not to mention them sometimes as well as $|\beta| > 1$ will be taken large enough several times without mentioning.

Lemma 2.7. *Under the above notations and the assumption* (2.6)*, we have $i\mathbb{R} \subseteq \rho(\mathcal{A})$.*

Proof. Since $\mathcal{A}$ is closed and $D(\mathcal{A})$ is compactly embedded in $\mathcal{H}$, then according to Theorem 2.6 (reminded above) the spectrum $\sigma(\mathcal{A}) = \mathbb{C}/\rho(\mathcal{A})$ has only eigenvalues. Let us suppose that there exists a pure imaginary eigenvalue $i\beta \neq 0$ with corresponding non-null eigenfunction $U = (\varphi, \Phi, \psi, \Psi, \theta)$. Then, from (2.17) with $F = 0$, the dissipativity (2.11) and condition (2.6), we have $\theta \equiv 0$, and going back to the system (2.18)–(2.22) with $F = 0$ one gets $U \equiv 0$, which is a contradiction. Therefore, $i\mathbb{R} \subseteq \rho(\mathcal{A})$. □

Lemma 2.8. *Under the above notations and the assumption* (2.6)*, there exists a constant $C > 0$ such that*

$$\|\theta_x\|_{L^2}^2 \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{2.24}$$

Proof. Immediately from (2.17), (2.11) and then (2.23). □

To the next computations, we shall use some useful cut-off functions and a modified "cut-off" resolvent equation to obtain firstly some local estimates.

Auxiliary cut-off functions s_j, $j = 1, 2$. Let us consider $l_0 \in (0, l)$ and $\delta > 0$ arbitrary numbers such that $(l_0 - \delta, l_0 + \delta) \subset (0, l)$ and $s_1, s_2 \in C^2(0, l)$ are cut-off functions satisfying

$$\operatorname{supp} s_j \subset (l_0 - \delta/j, l_0 + \delta/j), \quad 0 \le s_j(x) \le 1,\ x \in (0, l), \tag{2.25}$$

and

$$s_j(x) = 1 \quad \text{for} \quad x \in [l_0 - \delta/(j+1), l_0 + \delta/(j+1)], \quad j = 1, 2. \tag{2.26}$$

The idea of the graphic for such cut-off functions is disposed in Figure 1 as follows.

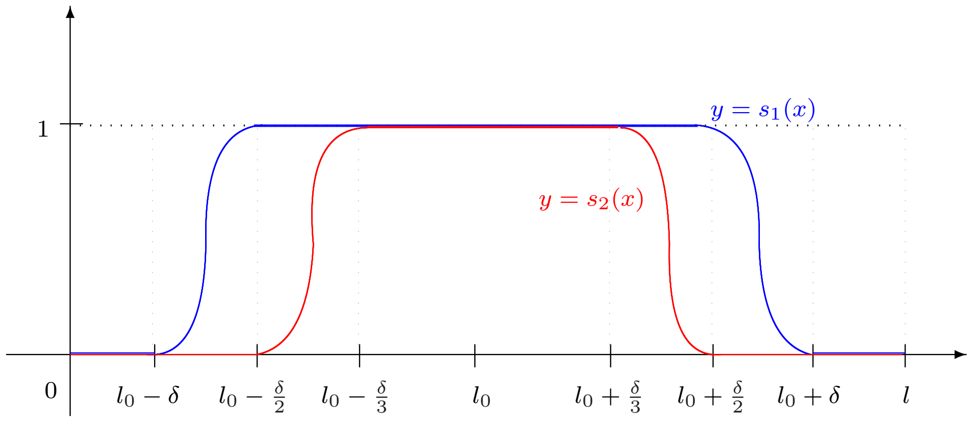

FIGURE 1 Geometric description of functions s_1 and s_2

Auxiliary resolvent equation. Given $U = (\varphi, \Phi, \psi, \Psi, \theta) \in D(\mathcal{A})$ solution of (2.17), we additionally introduce the following functions

$$\varphi^j = s_j\varphi, \quad \Phi^j = s_j\Phi, \quad \psi^j = s_j\psi, \quad \Psi^j = s_j\Psi, \quad \theta^j = s_j\theta, \quad j = 1, 2. \tag{2.27}$$

Then, for each $j = 1, 2$, we have from (2.18)–(2.22) that the above functions satisfy

$$i\beta\varphi^j - \Phi^j = s_j f_1, \tag{2.28}$$

$$i\beta\rho_1\Phi^j - \left(k\left(\varphi^j_x + \psi^j\right)\right)_x = s_j\rho_1 f_2 + V_j, \tag{2.29}$$

$$i\beta\psi^j - \Psi^j = s_j f_3, \tag{2.30}$$

$$i\beta\rho_2\Psi^j - \left(b\,\psi^j_x\right)_x + k\left(\varphi^j_x + \psi^j\right) + \left(m\,\theta^j\right)_x = s_j\rho_2 f_4 + Y_j, \tag{2.31}$$

$$i\beta\rho_3\theta^j - \left(c\,\theta^j_x\right)_x + m\Psi^j_x = s_j\rho_3 f_5 + \Theta_j, \tag{2.32}$$

where denote

$$\begin{aligned} V_j &= -\left(s_j' k\varphi\right)_x - s_j' k(\varphi_x + \psi), \\ Y_j &= -s_j' b\,\psi_x - \left(s_j' b\,\psi\right)_x + s_j' k\varphi + s_j' m\theta, \\ \Theta_j &= -s_j' c\,\theta_x - \left(s_j' c\,\theta\right)_x + s_j' m\Psi. \end{aligned} \tag{2.33}$$

Remark 2.9. It is worth mentioning that all functions defined in (2.27) (and their derivatives) vanish at the boundary $\{0, l\}$. Thus, whenever we compute integration by parts with these functions, we have that any pointwise boundary term vanishes as well. Therefore, in what follows we do not need to be worried about integration by parts and boundary pointwise terms. In addition, from (2.24) and definition of s_j we obtain

$$\|\theta^j_x\|^2_{L^2} \le C\|\theta_x\|^2_{L^2} \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{2.34}$$

and

$$\left\|V_j\right\|_{L^2}, \left\|Y_j\right\|_{L^2}, \le C\|U\|_{\mathcal{H}}, \quad \left\|\Theta_j\right\|_{L^2} \le C\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}, \tag{2.35}$$

for each $j = 1, 2$, and some constant $C > 0$.

Lemma 2.10. *Under the above notations and the assumption* (2.6)*, there exists a constant $C > 0$ such that*

$$\int_0^l \left(\left|\psi_x^1\right|^2 + \left|\Psi^1\right|^2\right) dx \le \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2 + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\left(\frac{1}{|\beta|^{1/2}}\|U\|_{\mathcal{H}}^{1/2}\|\theta_x\|_{L^2}^{1/2} + \|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}\right) + \frac{C}{|\beta|^{4/3}}\|U\|_{\mathcal{H}}^{4/3}\|\theta_x\|_{L^2}^{2/3} + \frac{C}{|\beta|^{5/2}}\|U\|_{\mathcal{H}}^2. \tag{2.36}$$

In particular, given $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\left\|\psi_x^1\right\|_{L^2}^2 + \left\|\Psi^1\right\|_{L^2}^2 \le \frac{\epsilon}{|\beta|^\nu}\|U\|_{\mathcal{H}}^2 + C_\epsilon|\beta|^\nu\|F\|_{\mathcal{H}}^2, \tag{2.37}$$

for $\nu = 0$ or $\nu = 2$, and $|\beta| > 1$ large enough.

Proof. Let us consider (2.28)–(2.35) with $j = 1$. First of all, from Equations (2.30) and (2.32) we note that

$$\rho_3\theta^1 - \left(\frac{c}{i\beta}\theta_x^1\right)_x + m\psi_x^1 = \frac{1}{i\beta}\left(m(s_1 f_3)_x + s_1\rho_3 f_5 + \Theta_1\right). \tag{2.38}$$

Taking the multiplier $b\overline{\psi_x^1}$ in (2.38) and performing integration by parts we get

$$\int_0^l mb\left|\psi_x^1\right|^2 dx = \int_0^l \left(\rho_3 b\theta^1\right)_x\overline{\psi^1}\,dx \underbrace{- \int_0^l \left(\frac{c}{i\beta}\theta_x^1\right)\left(b\overline{\psi_x^1}\right)_x dx}_{:=J_1} + \underbrace{\frac{1}{i\beta}\int_0^l b\Theta_1\overline{\psi_x^1}\,dx}_{:=J_2} + \frac{1}{i\beta}\int_0^l b\left(m(s_1f_3)_x + s_1\rho_3 f_5\right)\overline{\psi_x^1}\,dx. \tag{2.39}$$

Using (2.31) we rewrite J_1 as

$$J_1 = \int_0^l \left(\frac{c}{i\beta}\theta_x^1\right)\left(i\beta\rho_2\overline{\Psi^1} - k\left(\overline{\varphi_x^1 + \psi^1}\right) - \left(m\overline{\theta^1}\right)_x + s_1\rho_2\overline{f_4} + \overline{Y_1}\right) dx.$$

In addition, using the definition of Θ_1 in (2.33), integration by parts and Equation (2.22), then J_2 can be rewritten as follows

$$J_2 = -\frac{1}{i\beta}\int_0^l \left(s_1'mb\right)'\Psi^1\overline{\psi}\,dx + \frac{1}{i\beta}\int_0^l \left(c\left(s_1'b\right)'\theta_x\overline{\psi^1} - b\left(s_1'c\,\theta\right)_x\overline{\psi_x^1}\right) dx + \int_0^l s_1'b\rho_3\theta\overline{\psi^1}\,dx - \frac{1}{i\beta}\int_0^l s_1'b\rho_3 f_5\overline{\psi^1}\,dx.$$

Replacing these two last identities in (2.39) we deduce

$$\int_0^l mb\left|\psi_x^1\right|^2 dx = \int_0^l \rho_2 c\,\theta_x^1\overline{\Psi^1}dx - \frac{1}{i\beta}\int_0^l \left(s_1'mb\right)'\Psi^1\overline{\psi}\,dx + \int_0^l s_1'\rho_3 b\,\theta\overline{\psi^1}\,dx + J_3, \tag{2.40}$$

where

$$J_3 = \int_0^l \left(\rho_3 b\theta^1\right)_x\overline{\psi^1}\,dx + \frac{1}{i\beta}\int_0^l c\,\theta_x^1\left(-k\left(\overline{\varphi_x^1 + \psi^1}\right) - \left(m\overline{\theta^1}\right)_x + s_1\rho_2\overline{f_4} + \overline{Y_1}\right) dx + \frac{1}{i\beta}\int_0^l \left(c\left(s_1'b\right)'\theta_x\overline{\psi^1} - b\left(s_1'c\,\theta\right)_x\overline{\psi_x^1}\right) dx - \frac{1}{i\beta}\int_0^l s_1'b\rho_3 f_5\overline{\psi^1}\,dx + \frac{1}{i\beta}\int_0^l b\left(m(s_1f_3)_x + s_1\rho_3 f_5\right)\overline{\psi_x^1}\,dx.$$

From Equation (2.30), and estimates (2.34)–(2.35), we infer

$$|J_3| \le \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2,$$

for some constant $C > 0$. Going back to (2.40), using the Young inequality, Equation (2.20) and estimates (2.23)–(2.24), we arrive at

$$\int_0^l \left|\psi_x^1\right|^2 dx \le C\|\theta_x\|_{L^2}\left\|\Psi^1\right\|_{L^2} + \frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}\left\|\Psi^1\right\|_{L^2} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2. \tag{2.41}$$

On the other hand, taking the multiplier $-\overline{\psi^1}$ in (2.31), using integration by parts, Equation (2.30) and the definition of Y_1 in (2.33), we have

$$\int_0^l \rho_2\left|\Psi^1\right|^2 dx = \int_0^l \left(b\left|\psi_x^1\right|^2 + k\left|\psi^1\right|^2\right) dx + J_4 + J_5, \tag{2.42}$$

where

$$J_4 = \int_0^l s\left(\left(m\theta^1\right)_x - s_1'm\theta - s_1\rho_2 f_4\right)\overline{\psi}\, dx - \int_0^l s_1\rho_2\Psi^1\overline{f_3}\, dx,$$

$$J_5 = \int_0^l \left(k\varphi_x^1 + 2\left(s_1' b\psi\right)_x\right)\overline{\psi^1}\, dx - \int_0^l \left(\left(s_1' b\right)'\psi + s_1' k\varphi\right)\overline{\psi^1}\, dx.$$

From Equation (2.20) and estimate (2.34), it follows that

$$|J_4| \le \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2,$$

for some constant $C > 0$. In addition, using integration by parts, Equations (2.18), (2.20), (2.28), and estimate (2.41), we see that

$$\begin{aligned}
|J_5| &\le \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\left\|\psi_x^1\right\|_{L^2} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \\
&\le \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\left(\frac{1}{|\beta|^{1/2}}\|U\|_{\mathcal{H}}^{1/2}\|\theta_x\|_{L^2}^{1/2} + \|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}\right) \\
&\quad + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2}^{1/2}\left\|\Psi^1\right\|_{L^2}^{1/2} + \frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}^{3/2}\left\|\Psi^1\right\|_{L^2}^{1/2} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
\end{aligned}$$

Replacing these last two estimates in (2.42), taking into account the Poincaré inequality and (2.41), using proper Young inequalities and (2.23)–(2.24) we obtain

$$\begin{aligned}
\int_0^l \left|\Psi^1\right|^2 dx &\le \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2 \\
&\quad + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\left(\frac{1}{|\beta|^{1/2}}\|U\|_{\mathcal{H}}^{1/2}\|\theta_x\|_{L^2}^{1/2} + \|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}\right) + \frac{C}{|\beta|^{4/3}}\|U\|_{\mathcal{H}}^{4/3}\|\theta_x\|_{L^2}^{2/3} + \frac{C}{|\beta|^{5/2}}\|U\|_{\mathcal{H}}^2,
\end{aligned} \tag{2.43}$$

where we use $|\beta|^4 > |\beta|^{8/3} > |\beta|^{5/2} > 1$. Finally, adding (2.41) and (2.43), using again the Young inequality and (2.24), we obtain (2.36). In particular, applying once more the Young inequality with $\epsilon > 0$ and (2.24), and taking $|\beta| > 1$ large enough, then the estimate (2.37) follows. □

Lemma 2.11. *Under the above notations and the assumption* (2.6)*, there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
\int_0^l \left(\left|\varphi_x^2 + \psi^2\right|^2 + \left|\Phi^2\right|^2\right) dx &\le C\int_0^l \left|b\rho_1 - k\rho_2\right|\left|\Psi_x^2\right|\left|\Phi^2\right| dx + C\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}\left(\left\|\psi_x^1\right\|_{L^2} + \left\|\Psi^1\right\|_{L^2}\right) \\
&\quad + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2 + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}^2.
\end{aligned} \tag{2.44}$$

Proof. Let us consider (2.28)–(2.35) with $j = 2$. Multiplying (2.31) by $k\left(\overline{\varphi_x^2 + \psi^2}\right)$ and integrating over $(0, l)$ we get

$$
\begin{aligned}
\int_0^l k^2\left|\varphi_x^2+\psi^2\right|^2 dx ={}& \underbrace{\int_0^l \rho_2 k\,\Psi^2\left(\overline{i\beta\left(\varphi_x^2+\psi^2\right)}\right) dx}_{:=J_6} + \underbrace{\int_0^l k\left(b\,\psi_x^2\right)_x\left(\overline{\varphi_x^2+\psi^2}\right) dx}_{:=J_7} \\
&+ \int_0^l k\left(s_2\rho_2 f_4 - \left(m\,\theta^2\right)_x + s_2' m\theta\right)\left(\overline{\varphi_x^2+\psi^2}\right) dx \qquad (2.45)\\
&- \int_0^l k\left(s_2' b\,\psi_x + \left(s_2' b\,\psi\right)_x\right)\left(\overline{\varphi_x^2+\psi^2}\right) dx + \int_0^l s_2' k^2\varphi\left(\overline{\varphi_x^2+\psi^2}\right) dx,
\end{aligned}
$$

where we use the expression for Y_2 given in (2.33). Now let us rewrite the terms J_6 and J_7 as follows. Indeed, using Equations (2.28)–(2.30) and integration by parts we have

$$
J_6 = -\int_0^l \rho_2 k\,\Psi_x^2\overline{\Phi^2}\,dx - \int_0^l (\rho_2 k)'\Psi^2\overline{\Phi^2}\,dx + \int_0^l \rho_2 k\left|\Psi^2\right|^2 dx + \int_0^l \rho_2 k\,\Psi^2\left(\left(s_2\overline{f_1}\right)_x + s_2\overline{f_3}\right) dx,
$$

and

$$
J_7 = \int_0^l \rho_1 b\,\Psi_x^2\overline{\Phi^2}\,dx + \int_0^l \rho_1 b\,(s_2 f_3)_x\overline{\Phi^2}\,dx + \int_0^l s_2\rho_1 b\,\psi_x^2\overline{f_2}\,dx + \int_0^l b\,\psi_x^2\overline{V_2}\,dx.
$$

Replacing these last two equalities in (2.45) we obtain

$$
\int_0^l k^2\left|\varphi_x^2+\psi^2\right|^2 dx = \int_0^l (\rho_1 b - \rho_2 k)\,\Psi_x^2\overline{\Phi^2}\,dx + J_8 + J_9 + J_{10} + J_{11}, \qquad (2.46)
$$

where

$$
\begin{aligned}
J_8 ={}& \int_0^l \rho_2 k\,\Psi^2\left(\left(s_2\overline{f_1}\right)_x + s_2\overline{f_3}\right) dx + \int_0^l \rho_1 b\,(s_2 f_3)_x\overline{\Phi^2}\,dx + \int_0^l s_2\rho_1 b\,\psi_x^2\overline{f_2}\,dx \\
&+ \int_0^l k\left(s_2\rho_2 f_4 - \left(m\,\theta^2\right)_x + s_2' m\theta\right)\left(\overline{\varphi_x^2+\psi^2}\right) dx,\\
J_9 ={}& \int_0^l b\,\psi_x^2\overline{V_2}\,dx - \int_0^l k\left(s_2' b\,\psi_x + (s_2' b\,\psi)_x\right)\left(\overline{\varphi_x^2+\psi^2}\right) dx,\\
J_{10} ={}& \int_0^l \Psi^2\left(\rho_2 k\overline{\Psi^2} - (\rho_2 k)'\overline{\Phi^2}\right) dx,\\
J_{11} ={}& \int_0^l s_2' k^2\varphi\left(\overline{\varphi_x^2+\psi^2}\right) dx.
\end{aligned}
$$

It is easy to see that there exists a constant $C > 0$ such that

$$
|J_8| \le C\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
$$

Also, from conditions (2.25)–(2.26) on s_j, $j = 1, 2$, and the estimate (2.35), we obtain

$$
|J_9| + |J_{10}| \le C\|U\|_{\mathcal{H}}\left(\left\|\psi_x^1\right\|_{L^2} + \left\|\Psi^1\right\|_{L^2}\right).
$$

In addition, integrating by parts and by using Equations (2.18) and (2.20), we infer

$$
|\mathrm{Re}\, J_{11}| \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2 + \frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}^2.
$$

Going back to (2.46) and taking the real part, inserting these three last estimates and by using (2.23), we arrive at

$$\int_0^l \left|\varphi_x^2+\psi^2\right|^2 dx \le C\int_0^l |\rho_1 b-\rho_2 k|\left|\Psi_x^2\right|\left|\Phi^2\right| dx + C\|U\|_{\mathcal{H}}\left(\left\|\psi_x^1\right\|_{L^2}+\left\|\Psi^1\right\|_{L^2}\right)$$
$$+C\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|F\|_{\mathcal{H}}^2+\frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}^2. \tag{2.47}$$

On the other hand, taking the multiplier $-\overline{\varphi^2}$ in (2.29), performing integration by parts and using (2.28), we can write

$$\int_0^l \rho_1\left|\Phi^2\right|^2 dx = \int_0^l k\left|\varphi_x^2+\psi^2\right|^2 dx + J_{12}, \tag{2.48}$$

where

$$J_{12} = -\int_0^l s_2\rho_1 f_2\,\overline{\varphi^2}\,dx - \int_0^l s_2\rho_1\Phi^2\overline{f_1}\,dx - \int_0^l k\left(\varphi_x^2+\psi^2\right)\overline{\psi^2}\,dx - \int_0^l V_2\overline{\varphi^2}\,dx,$$

by adding and subtracting $\int_0^l k\left(\varphi_x^2+\psi^2\right)\overline{\psi^2}\,dx$. Then, from Equations (2.28) and (2.30), and the estimate (2.35), it follows that

$$|J_{12}| \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\beta|}\|U\|_{\mathcal{H}}^2,$$

for some constant $C>0$. Inserting this estimate in (2.48), by using (2.47) and (2.23) we obtain

$$\int_0^l \left|\Phi^2\right|^2 dx \le C\int_0^l |\rho_1 b-\rho_2 k|\left|\Psi_x^2\right|\left|\Phi^2\right| dx + C\|U\|_{\mathcal{H}}\left(\left\|\psi_x^1\right\|_{L^2}+\left\|\Psi^1\right\|_{L^2}\right)$$
$$+C\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|F\|_{\mathcal{H}}^2+\frac{C}{|\beta|}\|U\|_{\mathcal{H}}^2, \tag{2.49}$$

where $|\beta|^2>|\beta|>1$ is assumed. Therefore, combining (2.47) and (2.49) we conclude (2.44). □

Now is the precise moment where we apply the observability result given by Corollary A.2 (see Appendix A) to extend the localized estimates (Lemmas 2.10 and 2.11) to a global estimate on $(0,l)$ for the resolvent solution $U=(\varphi,\Phi,\psi,\Psi,\theta)$. More precisely, we have:

Proposition 2.12. *Under the above notations and the assumption* (2.6)*, there exists a constant $C>0$ such that*

$$\|U\|_{\mathcal{H}}^2 \le C\int_{l_0-\delta/2}^{l_0+\delta/2} |b\rho_1-k\rho_2|^2\left|\Psi_x^2\right|^2 dx + C\left(\left\|\psi_x^1\right\|_{L^2}^2+\left\|\Psi^1\right\|_{L^2}^2\right)+C\|F\|_{\mathcal{H}}^2. \tag{2.50}$$

Proof. Keeping in mind the conditions (2.25)–(2.26) on $s_j,\ j=1,2$, and remembering the definitions in (2.27), we deduce from (2.36) and (2.44) the following estimate

$$\int_{l_0-\delta/3}^{l_0+\delta/3}\left(|\psi_x|^2+|\Psi|^2+|\varphi_x+\psi|^2+|\Phi|^2\right)dx \le \widetilde{\Lambda}, \tag{2.51}$$

where

$$\widetilde{\Lambda} := C\int_0^l |b\rho_1-k\rho_2|\left|\Psi_x^2\right|\left|\Phi^2\right| dx + C\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|F\|_{\mathcal{H}}^2$$
$$+C\|U\|_{\mathcal{H}}\left(\left\|\psi_x^1\right\|_{L^2}+\left\|\Psi^1\right\|_{L^2}\right)+C\left(\left\|\psi_x^1\right\|_{L^2}^2+\left\|\Psi^1\right\|_{L^2}^2\right)+\frac{C}{|\beta|}\|U\|_{\mathcal{H}}^2.$$

Now, from Equations (2.18)–(2.21) one sees that (φ,Φ,ψ,Ψ) satisfies (A.1)–(A.4) with

$$g_1 := f_1,\quad g_2 := \rho_1 f_2,\quad g_3 := f_3,\quad g_4 := \rho_2 f_4-[m\theta]_x,$$

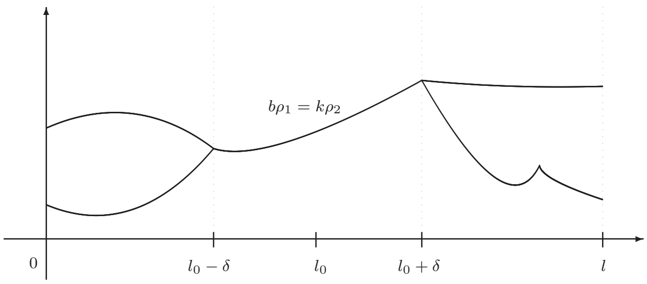

FIGURE 2 Geometric idea of the condition (2.53)

and also from (2.51) one has that (A.8) is verified with $a_1 = l_0 - \delta/3$ and $a_2 = l_0 + \delta/3$. Therefore, Corollary A.2 and (2.24) imply that

$$\int_0^l \left(|\psi_x|^2 + |\Psi|^2 + |\varphi_x + \psi|^2 + |\Phi|^2\right) dx \leq C\widetilde{\Lambda} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2. \tag{2.52}$$

Combining (2.24) with (2.52), and recalling (2.23), we obtain

$$\|U\|_{\mathcal{H}}^2 \leq C\widetilde{\Lambda} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2,$$

from where one concludes (2.50) after applying Hölder and Young's inequalities, (2.24) and choosing $|\beta| > 1$ large enough. □

3 | COMPLETION OF THE PROOF OF THEOREM 2.2 (EXPONENTIAL STABILITY)

The theoretical result for the conclusion of Theorem 2.2 is recalled in Theorem 2.4. Accordingly, the first part of the necessary condition (2.15) was proved in Lemma 2.7. Now, let $\epsilon > 0$ be given and let us choose $l_0 \in I$ and $\delta > 0$ such that $(l_0 - \delta, l_0 + \delta) \subset I$ as taken in (2.12). Thus,

$$b(x)\rho_1(x) - k(x)\rho_2(x) = 0 \quad \text{for all} \quad x \in (l_0 - \delta, l_0 + \delta). \tag{2.53}$$

See Figure 2.

From Proposition 2.12 and Lemma 2.10 (see (2.37) with $\nu = 0$) we have

$$\|U\|_{\mathcal{H}}^2 \leq \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2.$$

Therefore, taking $\epsilon > 0$ small enough, there exists a constant $C > 0$ such that

$$\|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}, \quad |\beta| \to \infty,$$

and from the resolvent equation (2.17) we conclude

$$\left\|\left(i\beta I_d - \mathcal{A}\right)^{-1} F\right\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}, \quad |\beta| \to \infty.$$

This shows the second property of condition (2.15). Hence, the conclusion of Theorem 2.2 follows from Theorem 2.4. □

4 | COMPLETION OF THE PROOF OF THEOREM 2.3 (POLYNOMIAL STABILITY)

The abstract result for the conclusion of the proof of Theorem 2.3 is based on the polynomial stability of bounded semigroups, as regarded in Theorem 2.5.

We begin by taking an arbitrary interval $(l_0 - \delta, l_0 + \delta) \subset (0, l)$ and $\epsilon > 0$. From Proposition 2.12, Equation (2.30), conditions (2.25)–(2.26) on s_j, $j = 1, 2$, and Lemma 2.10 (see (2.37) with $\nu = 0$ and $\nu = 2$) yield

$$\|U\|_{\mathcal{H}}^2 \leq C|\beta|^2\left\|\psi_x^1\right\|_{L^2}^2 + C\left(\left\|\psi_x^1\right\|_{L^2}^2 + \left\|\Psi^1\right\|_{L^2}^2\right) + C\|F\|_{\mathcal{H}}^2 \leq \epsilon\, C\|U\|_{\mathcal{H}}^2 + C_\epsilon|\beta|^4\|F\|_{\mathcal{H}}^2,$$

for some constant $C > 0$. Thus, choosing $\epsilon > 0$ small enough, there exists a constant $C > 0$ such that

$$\|U\|_{\mathcal{H}} \le C\,|\beta|^2 \|F\|_{\mathcal{H}}, \quad |\beta| \to \infty.$$

From the resolvent equation (2.17) we conclude

$$\left\| \left(i\beta I_d - \mathcal{A} \right)^{-1} F \right\|_{\mathcal{H}} \le C\,|\beta|^2 \|F\|_{\mathcal{H}}, \quad |\beta| \to \infty.$$

This shows the necessary assumption in the condition (2.16) with $\lambda = 2$ and so applying Theorem 2.5 one has

$$\left\| e^{\mathcal{A}t} \mathcal{A}^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \le \frac{C}{t^{1/2}}, \quad t \to \infty,$$

which in turn implies that the semigroup solution $U(t) = e^{\mathcal{A}t} U_0$ decays as

$$\|U(t)\|_{\mathcal{H}} \le \frac{C}{t^{1/2}} \|U_0\|_{D(\mathcal{A})}.$$

This proves (2.14) with $n = 1$ and initial data $U_0 \in D(\mathcal{A})$. The remaining decay rates in (2.14) follows from induction over $n \ge 2$ and the regularity of initial data. This completes the proof of Theorem 2.3. □

5 | HOMOGENEOUS THERMOELASTIC SYSTEM

In this section we are going to see another application of the observability inequality provided in Appendix A. More precisely, we shall prove that the rates of stability of the thermoelastic Timoshenko system depend on the equal wave speeds assumption, but are independent of boundary conditions in each related (exponential or polynomial) cases. The optimality of polynomial decay is also considered in some specific situations.

In order to simplify the notations, we consider hereafter the thermoelastic system (2.1)–(2.3) with constant (homogeneous) coefficients. In such a direction, we consider the next system

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in} \quad (0,l) \times \mathbb{R}^+, \tag{3.1}$$

$$\rho_2 \psi_{tt} - b\,\psi_{xx} + k(\varphi_x + \psi) + m\,\theta_x = 0 \quad \text{in} \quad (0,l) \times \mathbb{R}^+, \tag{3.2}$$

$$\rho_3 \theta_t - c\,\theta_{xx} + m\psi_{xt} = 0 \quad \text{in} \quad (0,l) \times \mathbb{R}^+, \tag{3.3}$$

with initial conditions

$$\varphi(\cdot,0) = \varphi_0,\ \varphi_t(\cdot,0) = \varphi_1,\ \psi(\cdot,0) = \psi_0,\ \psi_t(\cdot,0) = \psi_1,\ \theta(\cdot,0) = \theta_0 \ \text{ in } \ (0,l), \tag{3.4}$$

and Dirichlet or mixed Dirichlet–Neumann boundary conditions given by

$$\begin{array}{lll}
(\mathcal{B}C1) & \varphi(0,t) = \varphi(l,t) = \psi(0,t) = \psi(l,t) = \theta(0,t) = \theta(l,t) = 0, & t \ge 0,\\
(\mathcal{B}C2) & \varphi_x(0,t) = \varphi_x(l,t) = \psi(0,t) = \psi(l,t) = \theta(0,t) = \theta(l,t) = 0, & t \ge 0,\\
(\mathcal{B}C3) & \varphi(0,t) = \varphi(l,t) = \psi_x(0,t) = \psi_x(l,t) = \theta(0,t) = \theta(l,t) = 0, & t \ge 0,\\
(\mathcal{B}C4) & \varphi(0,t) = \varphi(l,t) = \psi(0,t) = \psi(l,t) = \theta_x(0,t) = \theta_x(l,t) = 0, & t \ge 0,\\
(\mathcal{B}C5) & \varphi_x(0,t) = \varphi_x(l,t) = \psi(0,t) = \psi(l,t) = \theta_x(0,t) = \theta_x(l,t) = 0, & t \ge 0.
\end{array} \tag{3.5}$$

It is worth pointing out that the problem (3.1)–(3.5) has the same characteristic as (2.1)–(2.5), being a particular case when one considers the boundary condition $(3.5)_{\mathcal{B}C1}$. In addition, as it will be clarified below, the main core in computations is not changed so that all results on stability (Theorems 2.2 and 2.3) remain unchanged. The only change falls on the spaces to approach different boundary conditions.

5.1 | Semigroup solution

To address all boundary conditions in (3.5), we consider the following phase spaces

$$\mathcal{H} = \begin{cases} H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) & \text{for } (3.5)_{\mathcal{B}C1}, \\ H_*^1(0,l) \times L_*^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) & \text{for } (3.5)_{\mathcal{B}C2}, \\ H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L^2(0,l) & \text{for } (3.5)_{\mathcal{B}C3}, \\ H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L_*^2(0,l) & \text{for } (3.5)_{\mathcal{B}C4}, \\ H_*^1(0,l) \times L_*^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L_*^2(0,l) & \text{for } (3.5)_{\mathcal{B}C5}, \end{cases}$$

under the same inner product and norm as defined in (2.7) and (2.8), respectively, where $H_*^1(0,l) = H^1(0,l) \cap L_*^2(0,l)$ and $L_*^2(0,l) = \left\{ u \in L^2(0,l);\ \frac{1}{l}\int_0^l u(x)\,dx = 0 \right\}$.

In the present case, problem (3.1)–(3.5) can be rewritten as follows

$$\begin{cases} U_t = \mathcal{A}_{\mathcal{B}C}\, U, \quad t > 0, \\ U(0) := U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0), \end{cases} \tag{3.6}$$

where $\mathcal{A}_{\mathcal{B}C} : D(\mathcal{A}_{\mathcal{B}C}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}_{\mathcal{B}C} U = \begin{pmatrix} \Phi \\ \dfrac{k}{\rho_1}(\varphi_x + \psi)_x \\ \Psi \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi) - \dfrac{m}{\rho_2}\theta_x \\ \dfrac{c}{\rho_3}\theta_{xx} - \dfrac{m}{\rho_3}\Psi_x \end{pmatrix} \tag{3.7}$$

for all $U = (\varphi, \Phi, \psi, \Psi, \theta)$ in the domain

$$D(\mathcal{A}_{\mathcal{B}C}) = \{ U \in \mathcal{H} \mid \varphi, \psi, \theta \in H^2(0,l) \text{ and } (\mathcal{B}Cn) \text{ is satisfied} \},$$

with $(\mathcal{B}Cn)$, $n = 1, \dots 5$, being given by

$$(\mathcal{B}Cn) := \begin{cases} \Phi, \Psi, \theta \in H_0^1(0,l) & \text{for} \quad (3.5)_{\mathcal{B}C1}, \\ \Psi, \varphi_x, \theta \in H_0^1(0,l),\ \Phi \in H_*^1(0,l) & \text{for} \quad (3.5)_{\mathcal{B}C2}, \\ \Phi, \psi_x, \theta \in H_0^1(0,l),\ \Psi \in H_*^1(0,l) & \text{for} \quad (3.5)_{\mathcal{B}C3}, \\ \Phi, \Psi, \theta_x \in H_0^1(0,l) & \text{for} \quad (3.5)_{\mathcal{B}C4}, \\ \Psi, \varphi_x, \theta_x \in H_0^1(0,l), \Phi \in H_*^1(0,l) & \text{for} \quad (3.5)_{\mathcal{B}C5}. \end{cases} \tag{3.8}$$

Under the above notations, the operator $\mathcal{A}_{\mathcal{B}C}$ defined in (3.7) is dissipative in $\mathcal{H}$ with

$$\mathrm{Re}\big(\mathcal{A}_{\mathcal{B}C} U, U\big)_{\mathcal{H}} = -c\|\theta_x\|_{L^2}^2 \le 0, \quad U \in D(\mathcal{A}_{\mathcal{B}C}), \tag{3.9}$$

independently of the boundary conditions $(\mathcal{B}C1)$–$(\mathcal{B}C5)$ in (3.5). Hence, the existence and uniqueness result to the abstract Cauchy problem (3.6) is stated analogously to Theorem 2.1. In conclusion, the system (3.1)–(3.5) is then well posed.

5.2 | Stability results

In the case of homogeneous coefficients the condition (2.12) reduces to $\frac{k}{\rho_1} = \frac{b}{\rho_2}$. Thus, the stability of the system (3.1)–(3.5) will depend of the number

$$\chi := \frac{\rho_1}{k} - \frac{\rho_2}{b}. \tag{3.10}$$

In the present section our main stability results read as follows:

Theorem 3.1 (Exponential Stability). *Let us assume in* (3.10) *that* $\chi = 0$. *Then, there exist constants* $C, \gamma > 0$, *independent of the initial data* $U_0 \in \mathcal{H}$, *such that the semigroup solution* $U(t) = e^{\mathcal{A}_{\mathcal{BC}}t}U_0$ *of* (3.6) *satisfies*

$$\|U(t)\|_{\mathcal{H}} \leq Ce^{-\gamma t}\|U_0\|_{\mathcal{H}}, \quad t > 0. \tag{3.11}$$

In other words, the homogeneous system (3.1)–(3.5) *is exponentially stable.*

Theorem 3.2 (Polynomial Stability). *Let us assume in* (3.10) *that* $\chi \neq 0$. *Then, there exists a constant* $C_m > 0$ *independent of the initial data* $U_0 \in D\big((\mathcal{A}_{\mathcal{BC}})^m\big)$, $m \geq 1$ *integer, such that the solution* $U(t) = e^{\mathcal{A}_{\mathcal{BC}}t}U_0$ *satisfies*

$$\|U(t)\|_{\mathcal{H}} \leq \frac{C_m}{t^{m/2}}\|U_0\|_{D((\mathcal{A}_{\mathcal{BC}})^m)}, \quad t \to +\infty. \tag{3.12}$$

In other words, the homogeneous system (3.1)–(3.5) *is polynomially stable with rate depending on the regularity of initial data.*

The proofs of Theorems 3.1 and 3.2 follow verbatim the same arguments as in the proofs of Theorems 2.2 and 2.3 provided along the Section 2.3.

5.3 | Optimality

In order to fix the ideas, let us deal with the problem (3.1)–(3.4) under the boundary condition $(3.5)_{\mathcal{BC}3}$. Then, by taking $\chi \neq 0$ and fixing $U_0 \in D\big(\mathcal{A}_{\mathcal{BC}}\big)$, we get from Theorem 3.2 that

$$\|U(t)\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}}\|U_0\|_{D(\mathcal{A}_{\mathcal{BC}})}, \quad t \to +\infty. \tag{3.13}$$

In what follows, we are going to conclude that the decay rate $t^{-1/2}$ in (3.13) is optimal. Such a statement will also allow us to conclude lack of exponential stability as well. More precisely, we have:

Theorem 3.3 (Optimality). *Under the above notations, the polynomial decay rate* $t^{-1/2}$ *in* (3.13) *is optimal. In particular, the system* (3.1)–(3.4) *with boundary condition* $(3.5)_{\mathcal{BC}3}$ *is not exponentially stable when* $\chi \neq 0$.

Proof. Let us consider $\chi \neq 0$ and fix $U_0 \in D\big(\mathcal{A}_{\mathcal{BC}}\big)$ for ($\mathcal{BC}3$) in (3.8). In order to prove the desired optimality, we shall argue by contraction.

Suppose the decay rate $t^{-1/2}$ in (3.13) can be improved to $t^{-1/(2-\nu)}$ for some $\nu \in (0, 2)$. Then there exists a constant $C > 0$ such that

$$\left\| e^{\mathcal{A}_{\mathcal{BC}}t}\big(\mathcal{A}_{\mathcal{BC}}\big)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/(2-\nu)}}, \quad t \to \infty,$$

and from Theorem 2.5 we get

$$\frac{1}{|\beta|^{2-\nu}}\left\| \big(i\beta I_d - \mathcal{A}_{\mathcal{BC}}\big)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad |\beta| \to \infty. \tag{3.14}$$

On the other hand, if we find sequences $(\beta_\mu)_{\mu\in\mathbb{N}} \subset \mathbb{R}^+$ and $(F_\mu)_{\mu\in\mathbb{N}} \subset \mathcal{H}$ satisfying $\lim_{\mu\to\infty}\beta_\mu = \infty$ and $\|F_\mu\|_{\mathcal{H}} \leq 1$, such that

$$\lim_{\mu\to+\infty} \frac{1}{|\beta_\mu|^{2-\nu}}\left\| \big(i\beta_\mu I_d - \mathcal{A}_{\mathcal{BC}}\big)^{-1} F_\mu \right\|_{\mathcal{H}} = +\infty, \tag{3.15}$$

then the contraction is reached through (3.14) and (3.15).

It remains to show (3.15). To this end, we assume (without loss of generality) that $l = \pi$ and take $F_\mu \in \mathcal{H}$ as

$$F_\mu(x) = \left(0,\ \frac{1}{\rho_1}\sin(\mu x),\ 0,\ \frac{1}{\rho_2}\cos(\mu x),\ \frac{1}{\rho_3}\sin(\mu x)\right).$$

It is easy to verify that $\|F_\mu\|_{\mathcal{H}} \le 1$ for all $\mu \ge 4$. In addition, the corresponding resolvent equation

$$\left(i\beta I_d - \mathcal{A}_{BC}\right)U_\mu = F_\mu \quad \Leftrightarrow \quad U_\mu = \left(i\beta I_d - \mathcal{A}_{BC}\right)^{-1}F_\mu \tag{3.16}$$

can be rewritten in terms of its components as follows

$$\begin{aligned} i\beta\varphi - \Phi &= 0, \\ i\beta\rho_1\Phi - k(\varphi_x + \psi)_x &= \sin(\mu x), \\ i\beta\psi - \Psi &= 0, \\ i\beta\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi) + m\theta_x &= \cos(\mu x), \\ i\beta\rho_3\theta - c\,\theta_{xx} + m\Psi_x &= \sin(\mu x), \end{aligned}$$

where we still denote $U_\mu := (\varphi, \Phi, \psi, \Psi, \theta)$ to simplify the notation. From the first and third equations of the above system one obtains the reduced system in terms of φ, ψ and θ

$$\begin{aligned} -\beta^2\rho_1\varphi - k(\varphi_x + \psi)_x &= \sin(\mu x), \\ -\beta^2\rho_2\psi - b\psi_{xx} + k(\varphi_x + \psi) + m\theta_x &= \cos(\mu x), \\ i\beta\rho_3\theta - c\,\theta_{xx} + i\beta m\psi_x &= \sin(\mu x). \end{aligned} \tag{3.17}$$

In view of the boundary condition $(3.5)_{\mathcal{B}C3}$ we look for solutions of (3.17) of the form

$$\varphi(x) = A\sin(\mu x), \quad \psi(x) = B\cos(\mu x), \quad \theta(x) = C\sin(\mu x), \quad x \in [0, \pi],$$

where $A = A_\mu$, $B = B_\mu$ and $C = C_\mu$ will be determined later. In this way, to solve (3.17) is equivalent to find a solution (A, B, C) for the algebraic system

$$\begin{aligned} \left(-\beta^2\rho_1 + k\mu^2\right)A + k\mu B &= 1, \\ k\mu A + \left(-\beta^2\rho_2 + b\mu^2 + k\right)B + m\mu C &= 1, \\ -im\beta\mu B + \left(i\beta\rho_3 + c\mu^2\right)C &= 1. \end{aligned} \tag{3.18}$$

We denote the matrix of coefficients in (3.18) by

$$M = \begin{pmatrix} p_1 & k\mu & 0 \\ k\mu & p_2 & m\mu \\ 0 & -im\beta\mu & p_3 \end{pmatrix} \quad \text{where} \quad \begin{cases} p_1 = -\beta^2\rho_1 + k\mu^2, \\ p_2 = -\beta^2\rho_2 + b\mu^2 + k, \\ p_3 = i\beta\rho_3 + c\mu^2, \end{cases} \tag{3.19}$$

are functions of β. Then, a simple calculation shows that

$$\det M = \left[p_1 p_2 - k^2\mu^2\right]p_3 + i\beta\mu^2 m^2 p_1.$$

In what follows we choose a sequence $\beta = \beta_\mu$ so that $\det M \neq 0$. Then we take

$$\beta_\mu = \sqrt{\frac{k}{\rho_1}\mu^2 - \frac{c_0}{\rho_1}} \quad \text{with} \quad c_0 = \frac{k\rho_1}{b\chi}, \quad \mu \in \mathbb{N}, \quad \mu^2 > \frac{c_0}{k}. \tag{3.20}$$

Note that c_0 is well-defined since we have $\chi \neq 0$ in (3.10). Moreover, $p_1 \equiv c_0$ and from the definition of p_2 in (3.19) along with the identities for β_μ, c_0 in (3.20), we can define c_1 as

$$c_1 = p_1 p_2 - k^2\mu^2 = c_0 p_2 - k^2\mu^2 = c_0 k + c_0^2 \frac{\rho_2}{\rho_1} = \frac{k\rho_1^2}{b\chi^2} > 0,$$

which implies, observing p_3 in (3.19), that

$$\det M = c_1 p_3 + i\beta_\mu \mu^2 m^2 c_0 = c_1 c\,\mu^2 + i\beta_\mu \big(c_1\rho_3 + \mu^2 m^2 c_0\big) \neq 0.$$

Thus, the solution of (3.18) is given by

$$\begin{aligned} A_\mu &= \frac{p_3[p_2 - k\mu] + km\mu^2 + i\beta_\mu \mu^2 m^2}{\det M}, \\ B_\mu &= \frac{p_1[p_3 - m\mu] - p_3 k\mu}{\det M}, \\ C_\mu &= \frac{p_1[p_2 + i\beta_\mu \mu m] - k^2\mu^2 - i\beta_\mu \mu^2 mk}{\det M}. \end{aligned}$$

Besides, from the choice of β_μ in (3.20) one sees that $\beta_\mu \approx \sigma_0\mu,\ \sigma_0 > 0$, when $\mu \to +\infty$, and regarding the definitions of p_2 and p_3 in (3.19) we deduce $|A_\mu| \approx \sigma_1\mu,\ \sigma_1 > 0$, when $\mu \to +\infty$. Keeping in mind that $\Phi(x) = i\beta_\mu\varphi(x) = i\beta_\mu A_\mu \sin(\mu x),\ x \in [0,\pi]$, then

$$\|U_\mu\|_{\mathcal{H}}^2 \geq \rho_1 \int_0^\pi |\Phi(x)|^2\,dx = \rho_1|\beta_\mu|^2|A_\mu|^2 \int_0^\pi \sin^2(\mu x)\,dx = \frac{\rho_1\pi}{2}|\beta_\mu|^2|A_\mu|^2, \tag{3.21}$$

which implies

$$|\beta_\mu|^{\nu-2}\|U_\mu\|_{\mathcal{H}} \geq \sqrt{\frac{\rho_1\pi}{2}}|\beta_\mu|^{\nu-1}|A_\mu| \approx \sigma_2\,\mu^\nu,\ \ \sigma_2 > 0,\ \text{ when } \mu \to +\infty. \tag{3.22}$$

Therefore, from (3.16) and (3.22),

$$\lim_{\mu\to+\infty} \frac{1}{|\beta_\mu|^{2-\nu}} \left\| \big(i\beta_\mu I_d - \mathcal{A}_{\mathcal{BC}}\big)^{-1} F_\mu \right\|_{\mathcal{H}} = \lim_{\mu\to+\infty} |\beta_\mu|^{\nu-2}\|U_\mu\|_{\mathcal{H}} = +\infty,$$

which proves (3.15). Hence, the optimality follows.

In particular, from (3.16) and (3.21) we also see that

$$\lim_{\mu\to+\infty} \left\| \big(i\beta_\mu I_d - \mathcal{A}_{\mathcal{BC}}\big)^{-1} F_\mu \right\|_{\mathcal{H}} = \lim_{\mu\to+\infty} \|U_\mu\|_{\mathcal{H}} = +\infty,$$

and from Theorem 2.4 the semigroup $\left\{e^{\mathcal{A}_{\mathcal{BC}}t}\right\}$ is not exponentially stable on $\mathcal{H}$.

This concludes the proof of Theorem 3.3. □

Remark 3.4. The optimality of the decay rate $t^{-1/2}$ provided by (3.13) was only proved for the boundary condition $(3.5)_{\mathcal{BC}3}$. However, such a result can also be extended to $(3.5)_{\mathcal{BC}5}$ with minor adjustments in the proof of Theorem 3.3, whereas some technical incompatibilities arise for the remaining boundary conditions. In conclusion, the homogeneous thermoelastic Timoshenko system (3.1)–(3.4) with boundary condition $(3.5)_{\mathcal{BC}3}$ or $(3.5)_{\mathcal{BC}5}$ is exponential stable if and only if $\chi = 0$. The optimality result for the boundary conditions $(3.5)_{\mathcal{BC}1}$, $(3.5)_{\mathcal{BC}2}$ and $(3.5)_{\mathcal{BC}4}$ is still open.

ACKNOWLEDGEMENT

M. A. Jorge Silva was partially supported by CNPq, Grants 441414/2014-1 and 402689/2012-7. T. F. Ma was partially supported by CNPq, Grant 312529/2018-0. J. E. Muñoz Rivera was partially supported by CNPq, Grant 308837/2014-2.

ORCID

M. A. Jorge Silva https://orcid.org/0000-0002-4806-886X

REFERENCES

[1] D. S. Almeida Júnior, M. L. Santos, and J. E. Muñoz Rivera, *Stability to weakly dissipative Timoshenko systems*, Math. Methods Appl. Sci. **36** (2013), 1965–1976.

[2] D. S. Almeida Júnior, M. L. Santos, and J. E. Muñoz Rivera, *Stability to 1-D thermoelastic Timoshenko beam acting on shear force*, Z. Angew. Math. Phys. **65** (2014), no. 6, 1233–1249.

[3] M. S. Alves, M. A. Jorge Silva, T. F Ma, and J. E. Muñoz Rivera, *Invariance of decay rate with respect to boundary conditions in thermoelastic Timoshenko systems*, Z. Angew. Math. Phys. **67** (2016), 70.

[4] M. S. Alves, M. A. Jorge Silva, T. F. Ma, and J. E. Muñoz Rivera, *Non-homogeneous thermoelastic Timoshenko systems*, Bull. Braz. Math. Soc. (N. S.) **48** (2017), 461–484.

[5] F. Ammar-Khodja, A. Benabdallah, J. E. Muñoz Rivera, and R. Racke, *Energy decay for Timoshenko systems of memory type*, J. Differential Equations **194** (2003), 82–115.

[6] F. Ammar-Khodja, S. Kerbal, and A. Soufyane, *Stabilization of the nonuniform Timoshenko beam*, J. Math. Anal. Appl. **327** (2007), 525–538.

[7] A. Borichev and Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. **347** (2010), 455–478.

[8] M. M. Cavalcanti, V. N. Domingos Cavalcanti, F. A. Falcão Nascimento, I. Lasiecka, and J. H. Rodrigues, *Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping*, Z. Angew. Math. Phys. **65** (2014), 1189–1206.

[9] M. Conti, F. Dell'Oro, and V. Pata, *Timoshenko systems with fading memory*, Dyn. Partial Differ. Equ. **10** (2013), 367–377.

[10] V. Danese, F. Dell'Oro, and V. Pata, *Stability analysis of abstract systems of Timoshenko type*, J. Evol. Equ. **16** (2016), 58–615.

[11] F. Dell'Oro and V. Pata, *On the stability of Timoshenko systems with Gurtin–Pipkin thermal law*, J. Differential Equations **257** (2014), 523–548.

[12] K. Engel and R. Nagel, *A short course on operator semigroups*, Springer, New York, 2006.

[13] L. H. Fatori, J. E. Muñoz Rivera, and R. N. Monteiro, *Energy decay to Timoshenko's system with thermoelasticity of type III*, Asymptot. Anal. **86** (2014), 227–247.

[14] H. D. Fernández Sare and R. Racke, *On the stability of damped Timoshenko systems: Cattaneo versus Fourier law*, Arch. Ration. Mech. Anal. **194** (2009), 221–251.

[15] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert space*, Trans. Amer. Math. Soc. **236** (1978), 385–394.

[16] F. L. Huang, *Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces*, Ann. Differential Equations **1** (1985), 43–56.

[17] M. A. Jorge Silva, T. F. Ma, and J. E. Muñoz Rivera, *Mindlin–Timoshenko systems with Kelvin–Voigt: analyticity and optimal decay rates*, J. Math. Anal. Appl. **417** (2014), 164–179.

[18] J. U. Kim and Y. Renardy, *Boundary control of the Timoshenko beam*, SIAM J. Control Optim. **25** (1987), 1417–1429.

[19] Z. Liu and S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC, Boca Raton, 1999.

[20] N. Mori, J. Xu, and S. Kawashima, *Global existence and optimal decay rates for the Timoshenko system: the case of equal wave speeds*, J. Differential Equations **258** (2015), 1494–1518.

[21] J. E. Muñoz Rivera and A. I. Ávila, *Rates of decay to non homogeneous Timoshenko model with tip body*, J. Differential Equations **258** (2015), no. 10, 3468–3490.

[22] J. E. Muñoz Rivera and R. Racke, *Mildly dissipative nonlinear Timoshenko systems – global existence and exponential stability*, J. Math. Anal. Appl. **276** (2002), 248–278.

[23] A. F. Neves, H. S. Ribeiro, and O. Lopes, *On the spectrum of evolution operators generated by hyperbolic systems*, J. Funct. Anal. **67** (1986) 320-344.

[24] P. Olsson and G. Kristensson, *Wave splitting of the Timoshenko beam equation in the time domain*, Z. Angew. Math. Phys. **45** (1994), 866–881.

[25] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1983.

[26] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc. **284** (1984), 847–857.

[27] C. A. Raposo, J. Ferreira, M. L. Santos, and N. N. O. Castro, *Exponential stability for the Timoshenko system with two weak dampings*, Appl. Math. Lett. **18** (2005), 535–541.

[28] B. Said-Houari and A. Kasimov, *Damping by heat conduction in the Timoshenko system: Fourier and Cattaneo are the same*, J. Differential Equations **255** (2013), 611–632.

[29] M. L. Santos, D. S. Almeida Júnior, and J. E. Muñoz Rivera, *The stability number of the Timoshenko system with second sound*, J. Differential Equations **253** (2012), 2715–2733.

[30] A. Soufyane, *Stabilisation de la poutre de Timoshenko*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 731–734.

[31] S. P. Timoshenko, *Vibration problems in engineering*, Van Nostrand, New York, 1955.

How to cite this article: Cardozo CL, Jorge Silva MA, Ma TF, Muñoz Rivera JE. Stability of Timoshenko systems with thermal coupling on the bending moment. *Mathematische Nachrichten*. 2019;1–19. https://doi.org/10.1002/mana.201800546

APPENDIX A: OBSERVABILITY INEQUALITY FOR SYSTEMS OF TIMOSHENKO TYPE

In order to make this paper more self-contained, we have introduced this short appendix that brings up some results already known in the literature. More precisely, we remember an observability inequality for systems of Timoshenko type in a general static framework. It constitutes a fundamental result to extend localized estimates to the whole bounded domain under consideration. Such achievement, and its consequences, were initially considered by the authors in [3,4,21].

We start by considering the following system:

$$i\beta u - v = g_1 \quad \text{in} \quad (0,l), \tag{A.1}$$

$$i\beta\rho_1 v - [k(u_x + w)]_x = g_2 \quad \text{in} \quad (0,l), \tag{A.2}$$

$$i\beta w - z = g_3 \quad \text{in} \quad (0,l), \tag{A.3}$$

$$i\beta\rho_2 z - [b\, w_x]_x + k(u_x + w) = g_4 \quad \text{in} \quad (0,l), \tag{A.4}$$

where $g_1, g_3 \in H_0^1(0,l)$, $g_2, g_4 \in L^2(0,l)$ and ρ_1, ρ_2, k, b satisfy (2.6). For a vector-valued function $V = (u, v, w, z)$ and $0 \leq a_1 < a_2 \leq l$, we use the notation $\|\cdot\|_{a_1,a_2}$ to stand for

$$\|V\|^2_{a_1,a_2} = \int_{a_1}^{a_2} \left(|v_x + w|^2 + |v|^2 + |w_x|^2 + |z|^2\right) dx.$$

Proposition A.1 ([4, Proposition 3.12]). *Under the above notations, let $V = (u, v, w, z)$ be a regular solution of* (A.1)–(A.4) *and suppose that $0 \leq a_1 < a_2 \leq l$. Then, there exist constants $C_0, C_1 > 0$ such that*

$$|u_x(a_j) + w(a_j)|^2 + |v(a_j)|^2 + |w_x(a_j)|^2 + |z(a_j)|^2 \leq C_0\|V\|^2_{a_1,a_2} + C_0\|G\|^2_{0,l}, \tag{A.5}$$

$$\|V\|^2_{a_1,a_2} \leq C_1\left[|u_x(a_j) + w(a_j)|^2 + |v(a_j)|^2 + |w_x(a_j)|^2 + |z(a_j)|^2\right] + C_1\|G\|^2_{0,l}, \tag{A.6}$$

for $j = 1, 2$, where $G = (g_1, g_2, g_3, g_4)$.

An important consequence of Proposition A.1 is the next corollary, which is the precise result we have used in the present paper.

Corollary A.2 ([4, Corollary 3.14]). *Let $V = (u, v, w, z)$ be a regular solution of system* (A.1)–(A.4). *If for some subinterval $(a_1, a_2) \subset (0,l)$ one has*

$$\|V\|^2_{a_1,a_2} = \int_{a_1}^{a_2} \left(|u_x + w|^2 + |v|^2 + |w_x|^2 + |z|^2\right) dx \leq \Lambda, \tag{A.7}$$

then there exists a constant $C > 0$ such that

$$\|V\|^2_{0,l} = \int_0^l \left(|u_x + w|^2 + |v|^2 + |w_x|^2 + |z|^2\right) dx \leq C\Lambda + C\|G\|^2_{0,l}, \tag{A.8}$$

where $G = (g_1, g_2, g_3, g_4)$.